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# Examples of Algebraic Surfaces with $q = 0$ and $p_g \leq 1$ which are Locally Hypersurfaces

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## § 1. Introduction

Algebraic surfaces with $q = p_g = 0$ have been studied through pluri-canonical mappings in various papers ([3, 5, 10, 11, 9, 12, 1, 2] ). The purpose of this note is to give examples of algebraic surfaces with $q = 0$ and $p_g \leq 1$ from the viewpoint of the singularity theory.

Let $\overline{M}$ be a compactification of an affine surface $M$ which is defined by

$$(1.1) \qquad g(\mathbf{w}) = w_1^a w_3^b + w_2^c w_3^d + w_3^e + 1 = 0$$

where $a > b$ , $c > d$ and

$$(1.2) \qquad a + b \geq c + d \geq e > 0.$$

This simple class of algebraic surfaces contains many interesting algebraic surfaces. The the fundamental group $\pi_1(\overline{M})$ is always a finite cyclic group ([7] ). In particular, the irregularity $q(\overline{M})$ is zero for such $\overline{M}$. In our previous paper [8], we have studied rational or K3-surfaces which are exceptional divisors of the resolutions of three dimensional Brieskorn singularities. In this paper we give five minimal surfaces of the above type with $p_g \leq 1$ which are not either rational or K3-surfaces. Though most of them are known surfaces, our method gives a different approach to them.

In § 2, we study a canonical way of the compactification $\overline{M}$ of M through the toroidal embedding theory.

In § 3, we study three algebraic surfaces $\overline{M}_1$, $\overline{M}_2$ and $\overline{M}_3$ with $q = p_g = 0$. $\overline{M}_1$ and $\overline{M}_3$ are known as an Enriques surface and a Godeaux surface. $\overline{M}_2$ is a minimal surface with $\pi_1(\overline{M}_2) = \mathbf{Z}/3\mathbf{Z}$, $e = 12$ and $K^2 = 0$ where K is a canonical divisor and e is the Euler characteristic.

In § 4, we study two minimal surfaces $\overline{M}_4$ and $\overline{M}_5$ with $q = 0$ and $p_g = 1$. $\overline{M}_4$ satisfies that $K^2 = 2$, $e = 22$ and $\pi_1(\overline{M}_4) = \mathbf{Z}/2\mathbf{Z}$. $\overline{M}_5$ is a simply connected surface with $K^2 = 1$ and $e = 23$. $\overline{M}_3$, $\overline{M}_4$ and $\overline{M}_5$ are surfaces of general type. There are systematical studies by Todorov for $\overline{M}_4$ and $\overline{M}_5$ ([11, 12] ).

## §2. Compactification

Unless otherwise stated, we use the same notations as in [7, 8] throughout this paper. Let $f_\Xi(z) = \sum_{i=1}^{4} z_1^{a_{i1}} \cdots z_4^{a_{i4}}$ be a homogeneous polynomial. We assume that $A_i = (a_{i1}, \ldots, a_{i4})$ $(i = 1, \ldots, 4)$ spin a three-simplex $\Xi$. Let $f(z) = f_\Xi(z) + \sum_{i=1}^{4} z_i^N$ for a sufficiently large N and let $V = f^{-1}(0)$. Then V has an isolated singular point at the origin and the Newton boundary $\Gamma(f)$ is non-degenerate. Let $\Gamma^*(f)$ be the dual Newton diagram and let $\Sigma^*$ be a simplicial subdivision. Let $\pi : \tilde{V} \to V$ be the associated resolution of V. For each strictly positive vertex Q of $\Sigma^*$ with $\dim \Delta(Q) \geq 1$, there is a corresponding exceptional divisor $E(Q)$ of the above resolution ([7] ). Let $P = {}^t(1,1,1,1)$. Then $\Delta(P) = \Xi$ and $E(P)$ is the surface in which we are interested. The birational class of $E(P)$ does not depend on either the choice of N or on $\Sigma^*$ but depends only on $f_\Xi(z)$. Let $P_1, \ldots, P_4$ be the vertices of $\Sigma^*$ which are adjacent to P and $\dim \Delta(P_i) \geq 2$. We assume that $\Delta(P_i) \cap \Xi$ is the triangle with vertices $A_j$ for $j \neq i$. We also assume that $\Sigma^*$ is canonical around P on each triangle $T(P, P_i, P_j)$ in the sense of [7]. The fundamental group $\pi_1(E(P))$ is a finite cyclic group by Theorem (7.3) of [7].

Let M be the affine algebraic surface in $\mathbf{C}^3$ which is defined by

$$(2.1) \qquad g(w) = w_1^a w_3^b + w_2^c w_3^d + w_3^e + 1 = 0$$

where $a > b$ and $c > d$ and

(2.2) $$a + b \geq c + d \geq e > 0.$$

As the homogeneous polynomial $f_{\Xi}(z)$, we take

(2.3) $$f_{\Xi}(z) = z_1^a z_3^b + z_2^c z_3^d z_4^h + z_3^e z_4^i + z_4^{a+b}$$

where

(2.4) $$a + b = c + d + h = e + i.$$

We will show the following.

**Theorem** (2.5). _The exceptional divisor $E(P)$ is a smooth compactification of $M$._

_Proof_. To prove the assertion, it suffices to show that there exists a three dimensional simplex $\sigma = (P, Q_1, Q_2, Q_3)$ in $\Sigma^*$ such that the defining equation of $E(P)$ in $\mathbf{C}_\sigma^3 = \{ y_{\sigma 0} = 0 \} \cap \mathbf{C}_\sigma^4$ is equal to $g(y_{\sigma 1}, y_{\sigma 2}, y_{\sigma 3}) = 0$. Let $P_1, \ldots, P_4$ be the vertices of $\Sigma$ which are adjacent to $P$ and $\dim \Delta(P_i) \geq 2$ as before. It is easy to see that $P_1 \equiv {}^t(1,0,0,0)$ and $P_2 \equiv {}^t(0,1,0,0)$ modulo $\mathbf{Z}\langle P\rangle$. We assume that $P_3 \equiv {}^t(0,\alpha,\beta,\gamma)$ modulo $\mathbf{Z}\langle P\rangle$. By the definition, $P_3$ satisfies the following.

(2.6) $$b\beta = c\alpha + d\beta + h\gamma = (a + b)\gamma < e\beta + i\gamma.$$

Note that

(2.7) $$\det (P, P_1, P_2) = 1$$

and

$$(2.8) \qquad \det(P, P_1, P_2, P_3) = \beta - \gamma.$$

Here $\beta - \gamma$ is strictly positive by the inequality of (2.6) and (2.4). Thus we can take $Q_1 = P_1$, $Q_2 = P_2$ and

$$(2.9) \qquad Q_3 = (P_3 + \delta P_1 + \varepsilon P_2 + \theta P) / (\beta - \gamma)$$

where $\delta$, $\varepsilon$ and $\theta$ are integers such that $0 \leqq \delta, \varepsilon, \theta < (\beta - \gamma)$ as in Lemma (3.8) of [7]. If we replace $P_i$ by $P_i' = P_i + n_i P$ for some integer $n_i$, $\delta$ and $\varepsilon$ do not change but only $\theta$ changes in (2.9). Thus the defining equation of $E(Q)$ in $\mathbf{C}^3_\sigma$ does not change. See also the argument below. Thus we may assume that $P_1 = {}^t(1,0,0,0)$ and $P_2 = {}^t(0,1,0,0)$ and $P_3 = {}^t(0,\alpha,\beta,\gamma)$. Then the integrity of $Q_3$ implies that

$$(2.10) \quad \delta + \theta \equiv \varepsilon + \alpha + \theta \equiv \beta + \theta \equiv 0 \quad \text{modulo} \quad \beta - \gamma.$$

Let

$$h(\mathbf{y}_\sigma) = y_{\sigma 1}^{a'} y_{\sigma 3}^{b'} + y_{\sigma 2}^{c'} y_{\sigma 3}^{d'} + y_{\sigma 3}^{e'} + 1 = 0$$

be the defining equation of $E(P)$ in $\mathbf{C}^3_\sigma$. Then we have

$$a' = P_1(A_1) - d(P_1) = a,$$

$$b' = Q_3(A_1) - d(Q_3) = \delta a / (\beta - \gamma),$$

$$c' = P_2(A_2) - d(P_2) = c,$$

$$d' = Q_3(A_2) - d(Q_3) = \varepsilon c / (\beta - \gamma),$$

$$e' = Q_3(A_3) - d(Q_3) = (P_3(A_3) - d(P_3)) / (\beta - \gamma).$$

By (2.4) and (2.6), we have the following equalities.

$$b(\beta-\gamma) = a\gamma \quad \text{and} \tag{2.11}$$

$$c(\gamma-\alpha) = d(\beta-\gamma). \tag{2.12}$$

Therefore we have

$$\begin{aligned} b' &= \delta a / (\beta - \gamma) \\ &\equiv \beta a / (\beta - \gamma) \quad \text{modulo } a \text{ by } (2.10) \\ &\equiv \gamma a / (\beta - \gamma) \quad \text{modulo } a \\ &\equiv b \quad \text{modulo } a \text{ by } (2.11). \end{aligned}$$

As $0 \leqq b' < a$ and $b < a$ by the definition, this implies $b' = b$. Similarly we have

$$\begin{aligned} d' &= \varepsilon c / (\beta - \gamma) \\ &\equiv (\beta - \alpha) c / (\beta - \gamma) \quad \text{modulo } c \text{ by } (2.10) \\ &\equiv (\gamma - \alpha) c / (\beta - \gamma) \quad \text{modulo } c \\ &\equiv d \quad \text{modulo } c \text{ by } (2.12). \end{aligned}$$

As $0 \leqq d' < c$ and $d < c$, we have that $d' = d$. Finally

$$e' = (P_3(A_3) - d(P_3)) / (\beta - \gamma) = e.$$

Thus we have shown that $h(\mathbf{w}) = g(\mathbf{w})$, which completes the proof.

Hereafter we denote $E(P)$ by $\widetilde{M}$. In §3 and §4, we study algebraic surfaces $\overline{M}$ with $p_g \leq 1$. The details of the calculation for $K^2$, $e(\overline{M})$ and $\pi_1(\overline{M})$, we refer to [7] and [8].

**Remark** (2.13). Let $\Xi'$ be the simplex in $\mathbf{R}^3$ with vertices $(a,0,b)$, $(0,c,d)$, $(0,0,e)$ and $(0,0,0)$. Let $\nu^1, \ldots, \nu^k$ be the other possible integral points in $\Xi'$. Let

$$g_t(\mathbf{w}) = g(\mathbf{w}) + \sum_{i=1}^{k} t_i \, \mathbf{w}^{\nu^i}$$

and let $M_t$ be defined by $g_t(\mathbf{w}) = 0$. Let $U$ be the Zariski open set which is defined by the union of $t \in \mathbf{C}^k$ such that $g_t(\mathbf{w})$ is globally non-degenerate in the sense of [6]. Then $\{M_t\}$ $(t \in U)$ can be compactified simultaneously with $M = M_0$ and the complex manifold $\hat{M}$ which is the union $\bigcup_{t\in U} \overline{M}_t$ gives a k-dimensional deformation of $\overline{M}$. We call $\{\mathbf{w}^{\nu^i}\}$ <u>the embedded monomials</u> of $g(\mathbf{w})$. All the numerical calculations for $\overline{M}$ which follow in §3 and §4 remain true for $\overline{M}_t$.

## § 3. Surfaces with $q = p_g = 0$.

In this section, we will study three minimal algebraic surfaces $\overline{M}_1$, $\overline{M}_2$ and $\overline{M}_3$ with $q = p_g = 0$. $\overline{M}_1$ is known as an Enriques surface and $\overline{M}_3$ is a Godeaux surface. $\overline{M}_2$ is a minimal surface with $\pi_1(\overline{M}_2) \cong \mathbf{Z}/3\mathbf{Z}$, $e(\overline{M}_2) = 12$ and $K^2 = 0$. Here $K$ is a canonical divisor and $e(\overline{M}_2)$ is the Euler characteristic.

(I) Let $M_1 = \{ g_1(\mathbf{w}) = 0 \}$ where

$$g_1(w) = w_1^4 w_3^3 + w_2^4 w_3^2 + w_3 + 1.$$

Then $f_\Delta(z) = z_1^4 z_3^3 + z_2^4 z_3^2 z_4 + z_3 z_4^6 + z_4^7$ is the corresponding homogeneous polynomial. We may take $P_3 = {}^t(0,1,7,3)$ and $P_4 = {}^t(0,-1,-6,-2)$. As det $(P, P_1, P_3)$ = det $(P, P_2, P_4) = 2$, we need two vertices $T_{13} = (P + P_1 + P_3) / 2$ on $T(P, P_1, P_3)$ and $T_{24} = (P_2 + P_4) / 2$ on $T(P, P_2, P_4)$ respectively. Here we are only considering vertices of $\Sigma^*$ which are adjacent to P. We denote the divisor $E(P) \cap E(P_i)$ in $E(P)$ by $C(P_i)$ etc. Let $\sigma = (P, P_1, P_2, R)$ be the fixed three-simplex of $\Sigma^*$ where $R = (3P_1 + P_2 + P_3 + P) / 4 = {}^t(1,1,2,1)$. Let $\omega$ be the meromorphic two form on $\overline{M}_1 = E(P)$ which is defined by

$$dy_{\sigma 1} \wedge dy_{\sigma 2} \wedge dy_{\sigma 3} / dg_1(y_\sigma)$$

on $\mathbf{C}^3_\sigma$ and $K = (\omega)$. By § 9 of [7], we get

$$K = 2C(P_4) + C(T_{24}) - 2C(P_3) - C(T_{13}), \tag{3.1}$$

$$K^2 = 0, \quad e(\overline{M}_1) = 12 \quad \text{and} \quad \pi(\overline{M}_1) \cong \mathbf{Z}/2\mathbf{Z}. \tag{3.2}$$

Let $p : \widetilde{M}_1 \to \overline{M}_1$ be the universal covering and let $\varphi_{34}$ be the rational function on $\overline{M}_1$ which is defined by $\pi^*(z_4 z_3^{-1})$. Then we have that

$$(\varphi_{34}) = 2K \tag{3.4}$$

Thus there is a rational function $\psi$ on $\widetilde{M}_1$ such that $\psi^2 = p^*\varphi_{34}$. Then it is easy to see that $\psi^{-1} p^*\omega$ is a nowhere vanishing two-form on $\widetilde{M}_1$. This implies that $\widetilde{M}_1$ is a K3-surface and $\overline{M}_1$ is called an Enriques surface. (See Griffiths

[4], P.541 for the standard way of the construction of a Enriques surface.)

$g_1(w)$ has 6 embedded monomials $\mathbf{w}^{\nu^i}$ where $\{\nu^i\}$ $(i=1,\dots,6)$ are $(0,1,1)$, $(0,2,1)$, $(1,0,1)$, $(1,2,2)$, $(2,0,2)$ and $(2,1,2)$.

(II) Let $M_2 = \{\ g_2(\mathbf{w}) = 0\ \} \subset \mathbf{C}^3$ where

(3.5) $$g_2(\mathbf{w}) = w_1^9 w_3^6 + w_2^3 w_3^2 + w_3 + 1$$

Then $f_\Delta(z) = z_1^9 z_3^6 + z_2^3 z_3^2 z_4^{10} + z_3 z_4^{14} + z_4^{15}$ and $P_3 = {}^t(0,0,5,2)$ and $P_4 = {}^t(0,-2,-14,-5)$. As $\det(P,P_1,P_4) = 3$, we need a vertex $T_{14} = (P_4 + P_1 + 2P)/3$ on $T(P,P_1,P_4)$. Let $\sigma = (P,P_1,P_2,R)$ where $R = (P_3 + 2P_1 + 2P_2 + P)/3$. Then we have

(3.6) $$K = 7C(P_4) + 2C(T_{14}) - 2C(P_3), \quad K^2 = 0,$$

(3.7) $$e(\overline{M}_2) = 12 \quad \text{and} \quad \pi_1(\overline{M}_2) \cong \mathbf{Z}/3\mathbf{Z}.$$

As $(\varphi_{34}) = 9C(P_4) - 3C(P_3) + 3C(T_{14})$, $3K$ is linearly equivalent to $3C(P_4)$. This easily proves that $\overline{M}_2$ is minimal.

$g_2(\mathbf{w})$ has 10 embedded monomials $\mathbf{w}^{\nu^i}$ where $\{\nu^i\}$ ( $i = 1,\dots,10$ ) are $(1,0,1)$, $(2,0,2)$, $(3,0,2)$, $(4,0,3)$, $(6,0,4)$, $(0,1,1)$, $(2,1,2)$, $(3,1,3)$, $(5,1,4)$ and $(1,2,2)$.

(III) Let $M_3 = \{g_3(\mathbf{w}) = 0\ \}$ where

(3.8) $$g_3(\mathbf{w}) = w_1^5 w_3^3 + w_2^5 w_3^2 + w_3 + 1.$$

Then $f_\Delta(z) = z_1^5 z_3^3 + z_2^5 z_3^2 z_4 + z_3 z_4^7 + z_4^8$ and $P_3 = {}^t(0,1,8,3)$ and $P_4 = {}^t(0,-1,-7,-2)$. Let $\sigma = (P, P_1, P_2, R)$ where $R = (P_3 + 3P_1 + 2P_2 + 2P)/5$. Then we have

(3.9) $$K = 2C(P_4) - C(P_3), \quad K^2 = 1,$$

(3.10) $$e(\overline{M}_3) = 11 \quad \text{and} \quad \pi_1(\overline{M}_3) \cong Z/5Z.$$

As $3K \sim C(P_4) + 2C(P_3)$, $\overline{M}_3$ is minimal by Lemma (4.23) of [8]. $\overline{M}_3$ is a Godeaux surface. See [10, 5]. $\overline{M}_3$ is isomorphic to the surface in Example (7.12) of [7].

$g_3(\mathbf{w})$ has 8 embedded monomials $\mathbf{w}^{\nu^i}$ where $\{\nu^i\}$ $(i=1,\ldots,8)$ are $(1,0,1)$, $(3,0,2)$, $(0,1,1)$, $(1,1,1)$, $(2,1,2)$, $(0,2,1)$, $(2,2,2)$ and $(1,3,2)$. As 8 is the dimension of the moduli space of the Godeaux surface ([5] ), it is possible that our deformation is complete. We do not discuss this in this paper.

## §4. Surfaces with $q = 0$ and $p_g = 1$

In this section, we will study three minimal surfaces $\overline{M}_4$, $\overline{M}_5$ and $\overline{M}_6$ with $q = 0$ and $p_g = 1$.

**(IV)** Let $M_4 = \{ g_4(\mathbf{w}) = 0 \}$ where

(4.1) $$g_4(\mathbf{w}) = w_1^8 w_3^3 + w_2^4 w_3^2 + w_3 + 1.$$

Then $f_\Delta(z) = z_1^8 z_3^3 + z_2^4 z_3^2 z_4^5 + z_3 z_4^{10} + z_4^{11}$ and $P_3 = {}^t(0,-1,11,3)$ and $P_4 = {}^t(0,0,-5,-1)$. We need three vertices $T_{13}^1$, $T_{13}^2$ and $T_{13}^3$ on $T(P,P_1,P_3)$ where $T_{13}^1 = (P_3 + 3P_1 + P)/4$ and etc.. Let $\sigma = (P, P_1, P_2, R)$ where

$R = (P_3 + 3P_1 + 4P_2 + 5P) / 8$. Then we have

(4.2) $$K = C(P_4), \quad K^2 = 2,$$

(4.3) $$e(\overline{M}_4) = 22 \quad \text{and} \quad \pi_1(\overline{M}_4) \cong Z/2Z.$$

Thus $p_g = 1$ and $\overline{M}_4$ is minimal. It is known that there is an algebraic surface S with $q = p_g = 0$ and $\pi_1(S) \cong Z/4Z$ ([10] ). We do not know whether our surface $\overline{M}_4$ is the double cover of such a surface S or not.

$g_4(\mathbf{w})$ has 11 embedded monomials $\mathbf{w}^{\nu^i}$ where $\{\nu^i\}$ ( i = 1,..., 11 ) are (1,0,1), (2,0,1), (4,0,2), (5,0,2), (0,1,1), (3,1,2), (4,1,2),(0,2,1), (2,2,2) and (1,3,2).

**(V)** Let $M_5 = \{ g_5(\mathbf{w}) = 0 \}$ where

(4.7) $$g_5(\mathbf{w}) = w_1^6 w_3^4 + w_2^3 + w_3^2 + 1.$$

Then $f_\Delta(z) = z_1^6 z_3^4 + z_2^3 z_4^7 + z_3^2 z_4^8 + z_4^{10}$ and $P_3 = {}^t(0,2,5,2)$ and $P_4 = {}^t(0,-3,-4,-1)$. We need two vertices $T_{13}^1$ and $T_{13}^2$ on $T(P,P_1,P_3)$ where $T_{13}^1 = (P_3 + 2P_1 + P) / 3$. We take $\sigma = (P,P_1,P_2,T_{13}^1)$ and by an easy calculation, we have

(4.8) $$K = C(P_4), \quad K^2 = 1,$$

(4.9) $$e(\overline{M}_5) = 23 \quad \text{and} \quad \pi_1(\overline{M}_5) = \{1\}.$$

$g_5(\mathbf{w})$ has 14 embedded monomials which correspond to (0,0,1), (1,0,1), (1,0,2), (2,0,2), (3,0,2), (3,0,3), (4,0,3), (0,1,0), (0,1,1), (1,1,1), (2,1,2), (3,1,2), (0,2,0) and (1,2,1). There are beautiful studies by Todorov for $\overline{M}_4$ and

$\overline{M}_5$ in [11, 12].

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